

COMMENT ON MAGNETIC MONOPOLE EXCITATIONS IN SPIN ICE

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Abstract: It has been proposed recently [3] that excitations in Spin Ice can be of the form of magnetic monopoles that does not obey the Dirac Quantization Condition. It is also well known [5] that the above scenario leads to non-associativity among translation generators. It will be interesting to see how the monopole picture in Spin Ice survives in the light of the latter observation.

Magnetic monopoles have become an enigma ever since Dirac [1] argued that the presence of only one of them can induce charge quantization through the product eg where e and g are respectively the strengths of electric and Magnetic Monopoles (MM). Search of MM in High Energy Physics has not led to any positive outcome. In recent years, experimental observation of MM and its theoretical explanation [2] in Condensed Matter systems has created quite a stir. However the singular structure observed here is in momentum space.

In a recent paper, Castelnovo, Moessner and Sondhi [3] have proposed that some exotic properties of Spin Ice can be explained by postulating that the elementary excitations above the ground state are in the form of monopole-antimonopole pairs. In the approximation considered in [3] the strings connecting the opposite poles comprise of dumbbells of oppositely charged magnetic poles that replace each spin in the original system. These strings carry a small amount of energy and are in principle observable. Moreover, (and this is crucial to the present Comment), the Dirac Quantization Condition (DQC) is not imposed here and in fact the monopole charge can be varied continuously by applying pressure on the system [3]. This can be contrasted to the conventional infinite Dirac string considered as a string of dipoles (see for example [4]) which, being an artifact, has to be unobservable and this is tied up with gauge invariance of the quantum system of a charge in the presence of a monopole, which eventually leads to the DQC.

A few years ago the problem of DQC was readdressed by Jackiw [5] in a lucid article (see also [6]). The beauty of the analysis [5] is that DQC is recovered in a *gauge invariant* way without taking recourse to a singular vector potential of Dirac string. On the other hand, the analysis is based on the simple fact that the quantum mechanics that we know today relies on the Hamiltonian formulation and lives in a Hilbert space. It is essential that the *Jacobi identities* involving the operators are satisfied and that the operators acting on the Hilbert space are *associative*. Jackiw's work [5] reveals that if one considers the semiclassical dynamics of a charge in an external MM field *without* imposing DQC, one is led to violation of Jacobi identities and non-associativity (among translation generators) in an essential way. From the very general nature of the arguments in [5], it appears that they can be relevant in the monopole excitation model of Spin Ice mentioned above [3].

Below we briefly reproduce the analysis of [5]. The gauge invariant dynamics of a massive (m) charge (e) in a non-dynamical magnetic field B is given in terms of the particle position operator $\mathbf{r}(\mathbf{t})$:

$$\dot{\mathbf{r}} = \mathbf{v} \quad ; \quad m\dot{\mathbf{v}} = \frac{e}{2c}(\mathbf{v} \times \mathbf{B} - \mathbf{B} \times \mathbf{v}), \quad (1)$$

keeping in mind the non-commutative nature of \mathbf{v} and $\mathbf{B}(\mathbf{r})$. The Hamiltonian, being independent of the magnetic field as it does not contribute to the particle energy, is

$$H = \frac{\pi^2}{2m} \quad ; \quad \pi \equiv m\mathbf{v}. \quad (2)$$

With the following non-trivial symplectic structure,

$$[r^i, r^j] = 0 \quad ; \quad [r^i, \pi^j] = i\hbar\delta^{ij} \quad ; \quad [\pi^i, \pi^j] = ie\frac{\hbar}{c}\epsilon^{ijk}B^k(\mathbf{r}), \quad (3)$$

one can reproduce the equations of motion (1) in a Hamiltonian framework:

$$\dot{\mathbf{r}} = \frac{i}{\hbar}[H, \mathbf{r}] = \frac{\pi}{m} \quad ; \quad \dot{\pi} = \frac{i}{\hbar}[H, \pi] = \frac{e}{2mc}(\pi \times \mathbf{B} - \mathbf{B} \times \pi). \quad (4)$$

However, Jacobi identity among the momenta yields,

$$\frac{1}{2}\epsilon^{ijk}[\pi^i, [\pi^j, \pi^k]] = \frac{e\hbar^2}{c}\nabla\cdot\mathbf{B} \quad (5)$$

which in fact vanishes for conventional magnetic fields $\nabla\cdot\mathbf{B} = 0$.

It needs to be stressed [5] that the non-canonical (or non-commutative) phase space structure (3) is forced on us if the Lorentz force law (1) is to be derived consistently in a Hamiltonian framework. One can contrast this with the normal case $\nabla\cdot\mathbf{B} = 0$ that leads to $\mathbf{B} = \nabla \times \mathbf{A}$. Now, with the identification $\pi = \mathbf{p} - \frac{e}{c}\mathbf{A}$ and the canonical phase space

$$[r^i, r^j] = 0 ; \quad [r^i, p^j] = i\hbar\delta^{ij} ; \quad [p^i, p^j] = 0,$$

it is trivial to derive non-commutative bracket $[\pi^i, \pi^j] = ie\frac{\hbar}{c}\epsilon^{ijk}B^k(\mathbf{r})$ and subsequently the Lorentz force law.

If one pursues further [5] one finds that for the translation generators,

$$T(\mathbf{a}) \equiv \exp(-\frac{i}{\hbar}\mathbf{a}\cdot\pi) ; \quad T^{-1}(\mathbf{a}) \mathbf{r} T(\mathbf{a}) = \mathbf{r} + \mathbf{a} \quad (6)$$

the Abelian composition law gets modified to

$$T(\mathbf{a}_1)T(\mathbf{a}_2) = \exp(-\frac{ie}{\hbar c}\Phi(\mathbf{r}; \mathbf{a}_1, \mathbf{a}_2))T(\mathbf{a}_1 + \mathbf{a}_2) ; \quad \Phi = (\mathbf{a}_1 \times \mathbf{a}_2)\cdot\mathbf{B} \quad (7)$$

This explicitly brings in to fore the loss of associativity with regard to translations,

$$(T(\mathbf{a}_1)T(\mathbf{a}_2))T(\mathbf{a}_3) = \exp(-\frac{ie}{\hbar c}\omega(\mathbf{r}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3))T(\mathbf{a}_1)(T(\mathbf{a}_2)T(\mathbf{a}_3)), \quad (8)$$

with the offending factor being,

$$\omega = \int d\mathbf{S}\cdot\mathbf{B} = \int dr \nabla\cdot\mathbf{B} \quad (9)$$

Clearly, ω measures the total magnetic flux coming out of the tetrahedron formed out of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, with one vertex at \mathbf{r} . Now DQC can come to the rescue by imposing the condition,

$$\int dr \nabla\cdot\mathbf{B} = 2\pi\frac{\hbar c}{e}N \quad (10)$$

since, for integer N , the exponential factor reduces to identity. Furthermore the above also requires that the sources of magnetic field have point support,

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3} ; \quad \nabla\cdot\mathbf{B} = 4\pi g\delta^3(\mathbf{r}). \quad (11)$$

Thus DQC $\frac{ge}{\hbar c} = \frac{N}{2}$ is restored. As for the Jacobi identity violation, it is not fully cured but is at least restricted only to the isolated points where the MMs are sitting [5].

Let us now return to the case at hand: Spin Ice. Loss of translation invariance will mean violation in the momentum conservation principle (as applied to a crystalline structure) and this can have direct experimental signatures. On the other hand, imposition of DQC will lead to a new quantization rule in the monopole magnitude of Spin Ice (see [3]). Either way, further study of the new material Spin Ice, both in theoretical and experimental context, is bound to reveal new physics in a fundamental way.

References

- [1] P.A.M.Dirac, Proc.R.Soc. A113, (1931) 60.
- [2] Z.Fang et.al., Science 302 (2003)92.
- [3] C.Castelnovo, R.Moessner and S.L.Sondhi, Nature 451, (2008) 42. (arXiv:0710.5515).
- [4] J.D.Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
- [5] R.Jackiw, Int.J.Mod.Phys. A19S1 (2004) 137 (arXiv:hep-th/0212058).
- [6] A.Berard, Y.Grandati and H.Mohrbach, Phys.Lett. A254 (1999) 133 (arXiv:physics/0004008).